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Global bifurcation and nodal solutions for homogeneous Kirchhoff type equations

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Abstract. In this paper, we shall study unilateral global bifurcation phenomenon for the following homogeneous Kirchhoff type problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right) u'' = \lambda u^3 + h(x,u,\lambda) & \text{in } (0,1),\\ u(0) = u(1) = 0. \end{cases}$$

As application of bifurcation result, we shall determine the interval of λ in which there exist nodal solutions for the following homogeneous Kirchhoff type problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right) u'' = \lambda f(x,u) & \text{in } (0,1),\\ u(0) = u(1) = 0, \end{cases}$$

where f is asymptotically cubic at zero and infinity. To do this, we also establish a complete characterization of the spectrum of a homogeneous nonlocal eigenvalue problem.

Keywords: bifurcation, spectrum, nonlocal problem, nodal solution, regularity results.
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1 Introduction

Consider the following problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right) u'' = \lambda u^3 + h(x,u,\lambda) & \text{in } (0,1),\\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where λ is a nonnegative parameter and $h : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{s\to 0} \frac{h(x,s,\lambda)}{s^3} = 0 \tag{1.2}$$

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uniformly for all $x \in (0,1)$ and λ on bounded sets.

The problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string [16]. Some important and interesting results can be found, for example, in [1, 4, 12, 13, 15, 19, 25]. Recently, there are many mathematicians studying the problem (1.1), see [5, 6, 8, 17, 20, 21, 22, 24, 26] and the references therein. A distinguishing feature of problem (1.1) is that the first equation contains a nonlocal coefficient $\int_0^1 |u'|^2\,dx$, and hence the equation is no longer a pointwise identity, which raises some essential difficulties to the study of this kind of problems. In particular, the bifurcation theory of [11, 23] does not work on it.

As shown in [3], the following problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)u'' = \lambda u^3 & \text{in } (0,1),\\ u(0)=u(1)=0 \end{cases} \tag{1.3}$$

possesses infinitely many eigenvalues $0<\mu_1<\mu_2<\cdots<\mu_k\to+\infty$, all of which are simple. The eigenfunction φ_k corresponding to μ_k has exactly $k-1$ simple zeros in $(0,1)$. Let S_k^+ denote the set of functions in $E := C_0^1[0,1]$ which have exactly $k-1$ interior nodal (i.e. non-degenerate) zeros in (0,1) and are positive near $x=0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. It is clear that S_k^+ and S_k^- are disjoint and open in E. Finally, let $\Phi_k^\pm = \mathbb{R}\times S_k^\pm$ and $\Phi_k = \mathbb{R}\times S_k$ under the product topology. The first main result of this paper is the following theorem.

Theorem 1.1. *The pair $(\mu_k, 0)$ is a bifurcation point of (1.1). Moreover, there are two distinct unbounded continua in $\mathbb{R}\times H_0^1(0,1)$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ emanating from $(\mu_k,0)$, such that $\mathscr{C}_k^\nu \subseteq \left(\{(\mu_k,0)\}\cup\Phi_k^\nu\right)$, $\nu\in\{+,-\}$.*

It is well known that the index formula of an isolated zero is very important in the study of bifurcation phenomena for semi-linear differential equations. However, problem (1.1) is nonlinear. In order to overcome this difficulty, we study the following auxiliary homogeneous eigenvalue problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)^{p/2} u'' = \lambda |u|^p u & \text{in } (0,1),\\ u(0)=u(1)=0, \end{cases} \tag{1.4}$$

where $p\in[0,2]$. We study the spectral structure, and establish an index formula via a suitable homotopic deformation from a general $p\in[0,2]$ to $p=0$ for problem (1.4). Let $\lambda_1(p)$ denote the first eigenvalue of (1.4). As shown in [9], $\lambda_1(p)>0$ is simple, isolated, the unique principal eigenvalue of (1.4), and is continuous with respect to p. Our second main result is the following theorem.

Theorem 1.2. *The set of all eigenvalues of (1.4) is formed by a sequence*

$$0<\lambda_1(p)<\lambda_2(p)<\cdots<\lambda_k(p)\to+\infty.$$

Every $\lambda_k(p)$ is simple, continuous with respect to p and the corresponding one dimensional space of solutions of (1.4) with $\lambda=\lambda_k(p)$ is spanned by a function having precisely k bumps in $(0,1)$. Each k-bump solution is constructed by the reflection and compression of the eigenfunction φ_1 associated with $\lambda_1(p)$.

Based on Theorem 1.1, we study the existence of nodal solutions for the following problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)u'' = \lambda f(x,u) & \text{in } (0,1),\\ u(0)=u(1)=0. \end{cases} \tag{1.5}$$

We assume that f satisfies the following conditions

(f1) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x,s)s > 0$ for all $x \in (0,1)$ and any $s \neq 0$.

(f2) there exist f_0, $f_\infty \in (0,+\infty)$ such that

$$\lim_{s\to 0^+} \frac{f(x,s)}{s^3} = f_0, \qquad \lim_{s\to +\infty} \frac{f(x,s)}{s^3} = f_\infty$$

uniformly with respect to all $x \in (0,1)$.

The last main theorem of this paper is the following result.

Theorem 1.3. *Assume that f satisfies (f1)–(f2). Then the pair $(\mu_k/f_0, 0)$ is a bifurcation point of (1.5) and there are two distinct unbounded continua in $\mathbb{R} \times H_0^1(0,1)$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, emanating from $(\mu_k/f_0, 0)$, such that $\mathscr{C}_k^\nu \subseteq \left(\{(\mu_k/f_0, 0)\} \cup \Phi_k^\nu\right)$ and links $(\mu_k/f_0, 0)$ to $(\mu_k/f_\infty, \infty)$.*

The rest of this paper is arranged as follows. In Section 2, we establish the spectrum of problem (1.4). In Section 3 and 4, we give the proofs of Theorem 1.1 and 1.3, respectively.

2 Spectrum of (1.4)

Let X be the usual Sobolev space $H_0^1(0,1)$ with the norm $\|u\| = \left(\int_0^1 |u'|^2\, dx\right)^{1/2}$. For any $\alpha \in (0,1]$, we use $C^\alpha[0,1]$ to denote all the real functions such that

$$\|u\|_\alpha := \sup_{x,y\in[0,1], x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < +\infty.$$

Firstly, we have the following regularity result.

Proposition 2.1. *Any weak solution $u \in X$ of problem (1.4) is also a classical solution, i.e., $u \in C^2[0,1]$ satisfying (1.4).*

Proof. Let u be a nontrivial weak solution of problem (1.4) and

$$f(x) = \frac{\lambda |u(x)|^p u(x)}{\|u\|^p}.$$

Note that

$$H_0^1(0,1) = \left\{u \in AC[0,1] : u' \in L^2(0,1) \text{ and } u(0) = u(1) = 0\right\}.$$

Then it is obvious that $f \in L^2(0,1)$, in fact continuous by the compact embedding $X \hookrightarrow C^{1/2}[0,1]$. According to the definition of weak solution, we have

$$-\left(\int_0^1 |u'|^2\, dx\right)^{\frac{p}{2}} u'' = \lambda |u|^p u$$

in the sense of distribution. It follows that

$$u'(x) = u'(0) - \int_0^x f(t)\, dt.$$

Note that

$$u(x) = \int_0^x u'(t)\, dt.$$

So, we have that

$$u(x) = \int_0^x \left(u'(0) - \int_0^t f(\tau)\, d\tau \right) dt = u'(0)\, x - \int_0^x \int_0^t f(\tau)\, d\tau\, dt.$$

Then, in view of $f \in C[0,1]$, we get that $u \in C^2[0,1]$ and satisfies (1.4). □

Lemma 2.2. *If (λ, u) is a solution of (1.4) and u has a double zero, then $u \equiv 0$.*

Proof. Let u be a solution of (1.4) and $x^* \in [0,1]$ be a double zero. If $\|u\| = 0$, the conclusion is obvious. Next, we assume that $\|u\| \neq 0$. We note that

$$u(x) = -\frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u\, d\tau\, ds.$$

Firstly, we consider $x \in [0, x^*]$. Then

$$\begin{aligned} |u(x)| &= \left| -\frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^s |u|^p u\, d\tau\, ds \right| \leq \left| \frac{\lambda}{\|u\|^p} \int_{x^*}^x \int_{x^*}^x |u|^p u\, d\tau\, ds \right| \\ &= \left| \frac{\lambda}{\|u\|^p} (x - x^*) \int_{x^*}^x |u|^p u\, d\tau \right| \\ &\leq \frac{\lambda}{\|u\|^p} \int_x^{x^*} |u|^{p+1}\, d\tau \leq \frac{\lambda \|u\|_\infty^p}{\|u\|^p} \int_x^{x^*} |u|\, d\tau \leq \lambda \int_x^{x^*} |u|\, d\tau. \end{aligned}$$

By the Gronwall–Bellman inequality [7, Lemma 2.2], we get $u \equiv 0$ on $[0, x^*]$. Similarly, we can get $u \equiv 0$ on $[x^*, 1]$ and the proof is completed. □

Lemma 2.3. *Each nontrivial solution (λ, u) of (1.4) has a finite number of zeros.*

Proof. Suppose, on the contrary, that u has a sequence zeros x_n. Since $[0,1]$ is compact, up to a subsequence, there exists $x_0 \in [0,1]$ such that $\lim_{n\to+\infty} x_n = x_0$. By the continuity of u, we have that $u(x_0) = \lim_{n\to+\infty} u(x_n) = 0$. So, we have that

$$u'(x_0) = \lim_{n\to+\infty} \frac{u(x_n) - u(x_0)}{x_n - x_0} = 0.$$

Thus, x_0 is a double zero of u. By Lemma 2.2, we get that $u \equiv 0$, which is a contradiction. □

Let J be a strict sub-interval of I. Let $\lambda_1(J)$ denote the first eigenvalue

$$\begin{cases} -\left(\int_0^1 |u'|^2\, dx \right)^{p/2} u'' = \lambda |u|^p u & \text{in } J, \\ u(x) = 0 & \text{on } \partial J, \end{cases}$$

where $p \in [0,2]$.

Lemma 2.4. *$\lambda_1(I)$ verifies the strict monotonicity property with respect to the domain I, i.e. if J is a strict subinterval of I, then $\lambda_1(I) < \lambda_1(J)$.*

Proof. Let φ_1 with $\|\varphi_1\| = 1$ be the eigenfunction of (1.4) on J corresponding to $\lambda_1(J)$, and denote by $\widetilde{\varphi}_1$ the extension by zero on I. Then we have that

$$\frac{1}{\lambda_1(J)} = \int_J |\varphi_1|^{p+2}\,dx = \int_I |\widetilde{\varphi}_1|^{p+2}\,dx < \sup_{u\in X,\|u\|=1} \int_0^1 |u|^{p+2}\,dx = \frac{1}{\lambda_1(I)}.$$

The last strict inequality holds from the fact that $\widetilde{\varphi}_1$ vanishes in $I \setminus J$ so cannot be an eigenfunction corresponding to the principal eigenvalue $\lambda_1(I)$. □

Proof of Theorem 1.2. Let φ_1 be a positive eigenfunction corresponding to $\lambda_1(p)$. It follows from the symmetry of (1.4) and Theorem 3.1 of [9] (or Theorem 2.4 of [18]) that $\varphi_1(x) = \varphi_1(1-x)$ for $x \in [0,1]$, i.e. φ_1 is even with respect to $1/2$. For any $k \geq 2$, set

$$\varphi_k(x) = \begin{cases} \varphi_1(kx), & x \in \left[0, \frac{1}{k}\right], \\ -\varphi_1(kx-1), & x \in \left[\frac{1}{k}, \frac{2}{k}\right], \\ \quad\vdots & \quad\vdots \\ (-)^k \varphi_1(kx-k+1), & x \in \left[\frac{k-1}{k}, 1\right]. \end{cases}$$

Then φ_k is an eigenfunction of (1.4) associated with the eigenvalue $\lambda_k(p) = k^{p+2}\lambda_1(p)$. Clearly, the continuity of $\lambda_1(p)$ implies that $\lambda_k(p)$ is continuous with respect to p.

On the other hand, let $u = u(x)$ be an eigenfunction of (1.4) associated with some eigenvalue $\lambda_* > \lambda_1(p)$. According to Theorem 3.1 of [9], u changes sign in $(0,1)$. Lemmas 2.2 and 2.3 imply that $u \in S_k$ for some $k \geq 2$. Without loss of generality, we may assume that $u'(0) > 0$. Let

$$0 < \tau_1 < \tau_2 < \cdots < \tau_{k-1} < 1$$

denote the zeros of u in $(0,1)$. Without loss of generality, we may assume that $\tau_1 \leq 1/k$. Applying Lemma 2.4 on $[0,1/k]$, we have that $\lambda_* \geq \lambda_k$. By Lemma 2 of [2], there exist integers p and q, $1 \leq p \leq k-1$, $1 \leq q \leq k-1$, such that

$$\tau_p \leq \frac{1}{q+1} < \frac{1}{q} \leq \tau_{p+1}.$$

Applying Lemma 2.4 on $[\tau_p, \tau_{p+1}]$, we have that $\lambda_* \leq \lambda_k$. So we have that $\lambda_* = \lambda_k$. Furthermore, if $\tau_1 < 1/k$, we have that $\lambda_* > \lambda_k$; if $\tau_1 > 1/k$, we have that $\lambda_* < \lambda_k$. Thus we have $\tau_1 = 1/k$ and $u = c_1\varphi_k(x)$ for $x \in [0,1/k]$. Similarly, we can obtain that $\tau_i = i/k$ and $u = c_i\varphi_k(x)$ for $x \in [(i-1)/k, i/k]$, $2 \leq i \leq k-1$. Let us normalize u as $u'(0) = \varphi_k'(0)$. It follows that $c_1 = 1$. Hence $\varphi_k'\left(\frac{1}{k}\right) = c_2\varphi_k'\left(\frac{1}{k}\right)$. So we have $c_2 = 1$. Similarly, one has $c_i = 1$ for all $3 \leq i \leq k-1$. Therefore, we have that $u(x) = \varphi_k(x)$, $x \in [0,1]$. □

3 Global bifurcation

Consider the following auxiliary problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right)^{p/2} u'' = f(x) & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{3.1}$$

for any $p \in [0,2]$ and a given $f \in X^*$. We have shown in [9] that problem (3.1) has a unique weak solution. Let us denote by $R_p(f)$ the unique weak solution of (3.1). Then $R_p : X^* \to X$

is a continuous operator. Since the embedding of $X \hookrightarrow L^\infty(0,1)$ is compact, the restriction of R_p to $L^1(0,1)$ is a completely continuous (i.e., continuous and compact) operator. From the obvious modification of Lemma 4.2 of [9], we can get the following compactness and continuity of the operator R_p with respect to p and f.

Lemma 3.1. *The operator $R : [0,2] \times L^1(0,1) \to L^\infty(0,1)$ defined by $R(p,f) = R_p(f)$ is completely continuous.*

Now, we consider (1.4) again. Clearly, u is a weak solution of (1.4) if and only if $u \in X$, $\lambda \in [0,+\infty)$ satisfy

$$u = R_p\left(\lambda |u|^p u\right) = \lambda^{\frac{1}{p+1}} R_p\left(|u|^p u\right) := T_p^\lambda(u).$$

For any $u \in X$, we define

$$K_p(u) = |u|^p u.$$

Then we see that $K_p(u) \in L^1(0,1)$. We claim that $K_p : X \hookrightarrow L^1(0,1)$ is continuous. Assume that $u_n \to u$ in X. Since embedding $X \hookrightarrow C[0,1]$ is compact, we have $u_n \to u$ in $C[0,1]$. It follows that $u_n(x) \to u(x)$ for any $x \in [0,1]$. So, we have that $K_p(u_n) \to K_p(u)$ in $L^1(0,1)$. Since $R_p : L^1(0,1) \to X$ is a compact, we have that $T_p^\lambda = \lambda^{\frac{1}{p+1}} R_p \circ K_p : X \to X$ is completely continuous. Thus the Leray–Schauder degree

$$\deg_X\left(I - T_p^\lambda, B_r(0), 0\right)$$

is well-defined for arbitrary r-ball $B_r(0)$ and $\lambda \neq \lambda_k(p)$. It is well known that

$$\deg_X\left(I - T_0^\lambda, B_r(0), 0\right) = (-1)^\beta,$$

where β is the number of eigenvalues of problem (1.4) with $p = 0$ less than λ. As far as the general p, we can compute it through the deformation along p.

Proposition 3.2. *Let $r > 0$ and $\overline{p} \in [0,2]$. Then*

$$\deg_X\left(I - T_{\overline{p}}^\lambda, B_r(0), 0\right) = \begin{cases} 1, & \text{if } \lambda \in (0, \lambda_1(\overline{p})), \\ (-1)^k, & \text{if } \lambda \in (\lambda_k(\overline{p}), \lambda_{k+1}(\overline{p})). \end{cases}$$

Proof. If $\lambda \in (0, \lambda_1(\overline{p}))$, the conclusion has done in [9]. So we only need to prove the case $\lambda \in (\lambda_k(\overline{p}), \lambda_{k+1}(\overline{p}))$. Since $p \to \lambda_k(p)$ is continuous, we can define a continuous function $\chi : [0,2] \to \mathbb{R}$ such that $\lambda_k(p) < \chi(p) < \lambda_{k+1}(p)$ and $\lambda = \chi(\overline{p})$. Set

$$d(p) = \deg_X\left(I - T_p^{\chi(p)}, B_r(0), 0\right).$$

We shall show that $d(p)$ is constant in $[0,2]$.

Define $S_p : L^\infty(0,1) \to X$ by $S_p(u) = R_p(\chi(p)|u|^p u)$. We see that $S_p(u) = \chi^{\frac{1}{p+1}}(p) R_p \circ K_p(u)$, where $K_p(u) = |u|^p u$. By the definition of K_p, we can easily verify that $K_p : L^\infty(0,1) \to L^1(0,1)$ is continuous. Since $R_p : L^1(0,1) \to X$ is a compact, we get that $S_p : L^\infty(0,1) \to X$ is completely continuous. Also we have that $T_p^{\chi(p)} = S_p \circ i$ where $i : X \to L^\infty(0,1)$ is the usual inclusion. From Lemma 2.4 of [14], we obtain that

$$d(p) = \deg_{L^\infty}\left(I - i \circ S_p, \Omega_s, 0\right) \quad \text{for } p \in [0,2],$$

where Ω_s is any open bounded set in $L^\infty(0,1)$ containing 0. It is not difficult to verify that the operator $\varphi : [0,2] \times L^\infty(0,1) \to L^1(0,1)$ defined by $\varphi(p,u) = |u|^p u$ is continuous. This fact, the continuity of $\chi(p)$ and Lemma 3.1 imply that $(p,u) \mapsto R_p\left(\chi(p)|u|^p u\right) = (i \circ S_p)(u) : [0,2] \times L^\infty(0,1) \to L^\infty(0,1)$ is completely continuous. Since $\lambda_k(p) < \chi(p) < \lambda_{k+1}(p)$ for any $p \in [0,2]$, we have that $u - R_p\left(\chi(p)|u|^p u\right) \neq 0$ on $\partial\Omega_s$. The invariance of the Leray–Schauder degree under a compact homotopy follows that $d(p) \equiv$ *constant* for $p \in [0,2]$. So, $d\left(\overline{p}\right) = d(0) = (-1)^k$, as desired. □

In particular, we have the following corollary.

Corollary 3.3. *Let $r > 0$. Then*

$$\deg_X\left(I - T_2^\lambda, B_r(0), 0\right) = \begin{cases} 1, & \text{if } \lambda \in (0, \mu_1), \\ (-1)^k, & \text{if } \lambda \in (\mu_k, \mu_{k+1}), \end{cases}$$

where μ_k is the k-th eigenvalue of (1.3).

Clearly, the pair (λ, u) is a solution of (1.1) if and only if (λ, u) satisfies

$$u = R_2\left(\lambda u^3 + h(x,u,\lambda)\right) := G_\lambda(u).$$

It is easy to see that $G_\lambda : X \to X$ is completely continuous and $G_\lambda(0) = 0$, $\forall \lambda \in [0, +\infty)$. μ_k is the λ_k. Let X_0 be any complement of span $\{\varphi_k\}$ in X.

Theorem 3.4. *The pair $(\mu_k, 0)$ is a bifurcation point of* (1.1). *Moreover, there are two distinct continua in $\mathbb{R} \times X$, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$, consisting of the bifurcation branch $\mathscr{C}_k$ emanating from $(\mu_k, 0)$, which contain $\{(\mu_k, 0)\}$ and each of them satisfies one of the following non-excluding alternatives:*

1. *it is unbounded in $\mathbb{R} \times X$;*
2. *it contains a pair $\left(\mu_j, 0\right)$ with $j \neq k$;*
3. *it contains a point $(\lambda, y) \in \mathbb{R} \times (X_0 \setminus \{0\})$.*

Proof. We use the abstract bifurcation result of [10] to prove this theorem. An operator L defined on X is called homogeneous if $L(cu) = cL(u)$ for any $c \in \mathbb{R}$ and $u \in X$. It is not difficult to verify that $L(\lambda) := T_2^\lambda : X \to X$ is homogeneous and completely continuous. Let $\widetilde{h}(x,u,\lambda) = \max_{0 \le |s| \le u} |h(x,s,\lambda)|$ for all $x \in (0,1)$ and λ on bounded sets, then $\widetilde{h}$ is nondecreasing with respect to u and

$$\lim_{u \to 0^+} \frac{\widetilde{h}(x,u,\lambda)}{u^3} = 0. \tag{3.2}$$

Further it follows from (3.2) that

$$\frac{h(x,u,\lambda)}{\|u\|^3} \le \frac{\widetilde{h}(x,|u|,\lambda)}{\|u\|_\infty^3} \le \frac{\widetilde{h}\left(x, \|u\|_\infty, \lambda\right)}{\|u\|_\infty^3} \to 0 \quad \text{as } \|u\| \to 0 \tag{3.3}$$

uniformly for $x \in (0,1)$ and λ on bounded sets. Let

$$H(\lambda, u) = G_\lambda(u) - L(\lambda)u.$$

By (3.3), we can easily verify that $H : \mathbb{R} \times X \to X$ is completely continuous with $H = o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. Noting Corollary 3.3, the desired conclusions can be obtained by applying Theorem 1 of [10]. □

By an argument similar to that of Proposition 2.1, we can get the following regularity result.

Proposition 3.5. *Any weak solution $u \in X$ of problem (1.1) is also a classical solution, i.e., $u \in C^2(0,1) \cap C^{1,\alpha}[0,1]$ satisfying (1.1) and $u(0) = u(1) = 0$.*

Lemma 3.6. *If (λ, u) is a solution of (1.1) and u has a double zero, then $u \equiv 0$.*

Proof. Let u be a solution of (1.1) and $x^* \in [0,1]$ be a double zero. If $\|u\| = 0$, the conclusion is done. Next, we assume that $\|u\| \neq 0$. We note that

$$u(x) = \frac{-1}{\|u\|^2} \int_{x^*}^{x} \int_{x^*}^{s} \left(\lambda u^3 + h(x,u,\lambda)\right) \, d\tau \, ds.$$

Firstly, we consider $x \in [0, x^*]$. Then

$$\begin{aligned} |u(x)| &\leq \frac{1}{\|u\|^2} \int_{x}^{x^*} \left|\lambda u^3 + h(x,u,\lambda)\right| \, d\tau, \\ &\leq \frac{\|u\|_\infty^2}{\|u\|^2} \int_{x}^{x^*} \left(|\lambda| + \left|\frac{h(\tau,u(\tau),\lambda)}{u(\tau)}\right|\right) |u(\tau)| \, d\tau. \end{aligned}$$

In view of (1.2), for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$|h(x,s,\lambda)| \leq \varepsilon |s|$$

uniformly with respect to all $x \in (0,1)$ and fixed λ when $|s| \in [0, \delta]$. Hence,

$$|u(x)| \leq \int_{x}^{x^*} \left(|\lambda| + \varepsilon + \max_{s \in [\delta, \|u\|_\infty]} \left|\frac{h(\tau,s,\lambda)}{s^3}\right|\right) |u(\tau)| \, d\tau.$$

By the Gronwall–Bellman inequality [7], we get $u \equiv 0$ on $[0, x^*]$. Similarly, we can get $u \equiv 0$ on $[x^*, 1]$ and the proof is complete. □

Proof of Theorem 1.1. Lemma 3.1 of [10] implies that there exists a bounded open neighborhood $\mathscr{O}_k$ of $(\mu_k, 0)$ such that $\left(\mathscr{C}_k^\nu \cap \mathscr{O}_k\right) \subseteq \left(\Phi_k^\nu \cup \{(\mu_k, 0)\}\right)$ or $\left(\mathscr{C}_k^\nu \cap \mathscr{O}_k\right) \subseteq \left(\Phi_k^{-\nu} \cup \{(\mu_k, 0)\}\right)$. Without loss of generality, we assume that $\left(\mathscr{C}_k^\nu \cap \mathscr{O}_k\right) \subseteq \left(\Phi_k^\nu \cup \{(\mu_k, 0)\}\right)$.

Next, we show that $\mathscr{C}_k^\nu \subseteq \left(\Phi_k^\nu \cup \{(\mu_k, 0)\}\right)$. Suppose $\mathscr{C}_k^\nu \not\subseteq \left(\Phi_k^\nu \cup \{(\mu_k, 0)\}\right)$. Then there exists $(\mu, u) \in \mathscr{C}_k^\nu \cap \left(\mathbb{R} \times \partial S_k^\nu\right)$ such that $(\mu, u) \neq (\mu_k, 0)$ and $(\lambda_n, u_n) \to (\mu, u)$ with $(\lambda_n, u_n) \in \mathscr{C}_k^\nu \cap \left(\mathbb{R} \times S_k^\nu\right)$. Since $u \in \partial S_k^\nu$, by Lemma 3.6, $u \equiv 0$. Let $v_n := u_n / \|u_n\|$, then v_n should be a solution of the following problem

$$v = R_2\left(\lambda_n v^3 + \frac{h\left(x, u_n, \lambda_n\right)}{\|u_n(x)\|^3}\right). \tag{3.4}$$

By (3.3), (3.4) and the compactness of R_2 we obtain that for some convenient subsequence $v_n \to v_0 \neq 0$ as $n \to +\infty$. Now v_0 verifies the equation

$$-\int_0^1 \left|v'\right|^2 dx v'' = \mu v^3$$

and $\|v_0\| = 1$. Hence $\mu = \mu_j$, for some $j \neq k$. Hence $v_0 \in S_j$ which is an open set in X, and as a consequence for some n large enough, $u_n \in S_j$, and this is a contradiction. Thus, we have that

$$\mathscr{C}_k^\nu \subseteq \left(\Phi_k^\nu \cup \{(\mu_k, 0)\}\right).$$

Furthermore, by an argument similar to the above, we can easily show that $\mathscr{C}_k \cap (\mathbb{R} \times \{0\}) = \{(\mu_k, 0)\}$. So Theorem 1 of [10] implies that $\mathscr{C}_k$ is unbounded.

We claim that both $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ are unbounded. Introduce the following auxiliary problem

$$\begin{cases} -\left(\int_0^1 |u'|^2\,dx\right) u'' = \lambda u^3 + \widetilde{h}(x,u,\lambda) & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $\widetilde{h}$ is defined by

$$\widetilde{h}(x,u,\lambda) = \begin{cases} h(x,u,\lambda), & \text{if } u'(0) > 0, \\ -h(x,-u,\lambda), & \text{if } u'(0) < 0. \end{cases}$$

The previous argument shows that an unbounded continuum $\widetilde{\mathscr{C}_k}$ bifurcates from $(\mu_k, 0)$ and can be split into $\widetilde{\mathscr{C}_k^+}$ and $\widetilde{\mathscr{C}_k^-}$ with $\widetilde{\mathscr{C}_k^\nu}$ connected, $\widetilde{\mathscr{C}_k^\nu} \subseteq \left(\{(\mu_k,0)\} \cup \left(\mathbb{R} \times S_k^\nu\right)\right)$. It is easy to see that $\widetilde{\mathscr{C}_k^-} = -\widetilde{\mathscr{C}_k^+}$. It follows that both $\widetilde{\mathscr{C}_k^+}$ and $\widetilde{\mathscr{C}_k^-}$ are unbounded. It is clear that $\widetilde{\mathscr{C}_k^+} \subseteq \mathscr{C}_k^+$. Therefore $\mathscr{C}_k^+$ must be unbounded. A symmetric argument shows that $\mathscr{C}_k^-$ is also unbounded. □

4 Nodal solutions

In this section, we apply Theorem 1.1 to study the existence of nodal solutions for (1.5).

Proof of Theorem 1.3. Let $g : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x,s) = f_0 s^3 + g(x,s)$$

with

$$\lim_{s \to 0} \frac{g(x,s)}{s^3} = 0 \quad \text{uniformly with respect to all } x \in (0,1). \tag{4.1}$$

From (4.1), we can see that λg satisfies the assumptions of (1.2). Now, using Theorem 1.1, we have that there are two distinct unbounded continua, $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ emanating from $(\mu_k/f_0, 0)$, such that

$$\mathscr{C}_k^\nu \subset \left(\{(\mu_k/f_0, 0)\} \cup \Phi_k^\nu\right).$$

It is sufficient to show that $\mathscr{C}_k^\nu$ joins $(\mu_k/f_0, 0)$ to $(\mu_k/f_\infty, \infty)$. Let $(\xi_n, u_n) \in \mathscr{C}_k^\nu$ where $u_n \not\equiv 0$ satisfies $|\xi_n| + \|u_n\| \to +\infty$. Proposition 5.1 of [8] implies that (0,0) is the only solution of (1.5) for $\lambda = 0$, we have $\mathscr{C}_k^\nu \cap (\{0\} \times X) = \emptyset$. It follows that $\xi_n > 0$ for all $n \in \mathbb{N}$.

Next we show that u_n is one-signed in some interval $(\alpha, \beta) \subseteq (0,1)$ with $\alpha < \beta$. Let

$$0 < \tau(1,n) < \tau(2,n) < \cdots < \tau(k-1,n) < 1$$

denote the zeros of u_n in $(0,1)$. Let $\tau(0,n) = 0$ and $\tau(k,n) = 1$. Then, after taking a subsequence if necessary,

$$\lim_{n\to+\infty} \tau(l,n) = \tau(l,\infty), \qquad l \in \{0,1,\ldots,k\}.$$

We claim that there exists $l_0 \in \{0,1,\ldots,k\}$ such that

$$\tau(l_0, \infty) < \tau(l_0+1, \infty).$$

Otherwise, we have that

$$1 = \Sigma_{l=0}^{k-1}(\tau(l+1,n) - \tau(l,n)) \to \Sigma_{l=0}^{k-1}(\tau(l+1,\infty) - \tau(l,\infty)) = 0.$$

This is a contradiction. Let $(\alpha,\beta) \subset (\tau(l_0,\infty), \tau(l_0+1,\infty))$ with $\alpha < \beta$. For all n sufficiently large, we have $(\alpha,\beta) \subset (\tau(l_0,n), \tau(l_0+1,n))$. So u_n does not change its sign in (α,β).

We claim that there exists a constant M such that $\xi_n \in (0, M]$ for $n \in \mathbb{N}$ large enough. On the contrary, we suppose that $\lim_{n\to+\infty} \xi_n = +\infty$. Since $(\xi_n, u_n) \in \mathscr{C}_k^{\nu}$, it follows that

$$\|u_n\|^2 u_n'' + \xi_n a_n(x) u_n^3 = 0 \quad \text{in } (0,1),$$

where

$$a_n(x) = \begin{cases} \frac{f(x,u_n)}{u_n^3}, & \text{if } u_n(x) \neq 0, \\ f_0, & \text{if } u_n(x) = 0. \end{cases}$$

From (f1)–(f2), we can see that $\frac{f(x,u_n)}{u_n} \geq \sigma$ for some $\sigma > 0$ and all $x \in (0,1)$, $n \in \mathbb{N}$. So, we have that $\xi_n a_n(x) = +\infty$ for all $x \in (0,1)$. Applying Theorem 4.1 of [3] on $[\alpha,\beta]$ with $g(x) \equiv \mu_1$, we have that u_n must change its sign in (α,β) for n large enough. This is a contradiction.

Therefore, we get that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty.$$

Let $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x,s) = f_\infty s^3 + h(x,s)$$

with

$$\lim_{|s|\to+\infty} \frac{h(x,s)}{s^3} = 0, \qquad \lim_{|s|\to 0} \frac{h(x,s)}{s^3} = f_0 - f_\infty \quad \text{uniformly with respect to all } x \in (0,1).$$

Then (ξ_n, u_n) satisfies

$$u_n = R_2\left(\xi_n f_\infty u_n^3 + h(x,u_n)\right).$$

Dividing the above equation by $\|u_n\|$ and letting $\overline{u}_n = u_n / \|u_n\|$, we get that

$$\overline{u}_n = R_2\left(\xi_n f_\infty \overline{u}_n^3 + \frac{h(x,u_n)}{\|u_n\|^3}\right).$$

Let

$$\widetilde{h}(x,u) = \max_{0\leq|s|\leq u} |h(x,s)| \quad \text{for any } x \in (0,1),$$

then $\widetilde{h}$ is nondecreasing with respect to u. Define

$$\overline{h}(x,u) = \max_{u/2\leq|s|\leq u} |h(x,s)| \quad \text{for any } x \in (0,1).$$

Then we can see that

$$\lim_{u\to+\infty} \frac{\overline{h}(x,u)}{u^3} = 0 \quad \text{and} \quad \widetilde{h}(x,u) \leq \widetilde{h}\left(x, \frac{u}{2}\right) + \overline{h}(x,u).$$

It follows that

$$\limsup_{u\to+\infty} \frac{\widetilde{h}(x,u)}{u^3} \leq \limsup_{u\to+\infty} \frac{\widetilde{h}\left(x,\frac{u}{2}\right)}{u^3} = \limsup_{u/2\to+\infty} \frac{\widetilde{h}\left(x,\frac{u}{2}\right)}{8\left(\frac{u}{2}\right)^3}.$$

So we have

$$\lim_{u\to+\infty}\frac{\widetilde{h}(x,u)}{u^3}=0. \tag{4.2}$$

Further it follows from (4.2) that

$$\frac{h\left(x,u_n\right)}{\|u_n\|^3}\le\frac{\widetilde{h}\left(x,|u_n|\right)}{\|u_n\|^3}\le\frac{\widetilde{h}\left(x,\|u_n\|_\infty\right)}{\|u_n\|^3}\le c^3\frac{\widetilde{h}\left(x,c\,\|u_n\|\right)}{c^3\,\|u_n\|^3}\to 0\quad\text{as } n\to+\infty$$

uniformly for $x\in(0,1)$.

By the compactness of R_2 we obtain that

$$-\|\overline{u}\|^2\,\overline{u}''=\overline{\mu}f_\infty\overline{u}^3,$$

where $\overline{u}=\lim_{n\to+\infty}\overline{u}_n$ and $\overline{\mu}=\lim_{n\to+\infty}\xi_n$, again choosing a subsequence and relabeling it if necessary. It follows from $\overline{u}=\lim_{n\to+\infty}\overline{u}_n$ and the triangle inequality that $\|\overline{u}\|=\lim_{n\to+\infty}\|\overline{u}_n\|$. Since $\|\overline{u}_n\|\equiv 1$, we obtain that $\|\overline{u}\|=1$. It is clear that $\overline{u}\in\mathscr{C}_k^\nu$. Theorem 1.2 of [3] shows that $\overline{\mu}=\mu_k/f_\infty$. Therefore, $\mathscr{C}$ joins $(\mu_k/f_0,0)$ to $(\mu_k/f_\infty,\infty)$. □

From Theorem 1.3, we can easily get the following corollary.

Corollary 4.1. *Assume that f satisfies (f1)–(f2). Then for*

$$\lambda\in\left(\frac{\mu_k}{f_0},\frac{\mu_k}{f_\infty}\right)\cup\left(\frac{\mu_k}{f_\infty},\frac{\mu_k}{f_0}\right),$$

problem (1.5) possesses at least two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0, and u_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near 0.

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References

[1] A. Arosio, S. Pannizi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348**(1996), 305–330. https://doi.org/10.1090/S0002-9947-96-01532-2; MR1333386

[2] H. Berestycki, On some nonlinear Sturm–Liouville problems, *J. Differential Equations* **26**(1977), 375–390. https://doi.org/10.1016/0022-0396(77)90086-9; MR0481230

[3] X. Cao, G. Dai, Spectrum, global bifurcation and nodal solutions for Kirchhoff type equations, *Electron. J. Differential Equations* **2018**, No. 179, 10 pp. MR3891484

[4] M. M. Cavalcanti, V. N. Cavalcanti, J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation, *Adv. Differential Equations* **6**(2001), 701–730. MR1829093

[5] C. Chen, Y. Kuo, T. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, *J. Differential Equations* **250**(2011), No. 4, 1876–1908. https://doi.org/10.1016/j.jde.2010.11.017; MR2763559

[6] B. Cheng, X. Wu, Existence results of positive solutions of Kirchhoff type problems, *Nonlinear Anal.* **71**(2009), No. 10, 4883–4892. https://doi.org/10.1016/j.na.2009.03.06; MR2548720

[7] G. Dai, Bifurcation and admissible solutions for the Hessian equation, *J. Funct. Anal.* **273**(2017), 3200–3240. https://doi.org/10.1016/j.jfa.2017.08.001; MR3695892

[8] G. Dai, Eigenvalue, global bifurcation and positive solutions for a class of nonlocal elliptic equations, *Topol. Methods Nonlinear Anal.* **48**(2016), 213–233. https://doi.org/10.12775/TMNA.2016.043; MR3561429

[9] G. Dai, Some global results for a class of homogeneous nonlocal eigenvalue problems, *Commun. Contemp. Math.* **21**(2019), No. 3, 1750093, 23 pp. https://doi.org/10.1142/S0219199717500936; MR3947059

[10] G. Dai, Z. Feng, Unilateral global bifurcation for eigenvalue problems with homogeneous operator, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **29**(2019), No. 6, 1950084, 15 pp. https://doi.org/10.1142/S0218127419500846; MR3976459

[11] E. N. Dancer, On the structure of solutions of non-linear eigenvalue problems, *Indiana Univ. Math. J.* **23**(1974), 1069–1076. https://doi.org/10.1512/iumj.1974.23.23087; MR0348567

[12] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.* **108**(1992), 247–262. https://doi.org/10.1007/BF02100605; MR1161092

[13] P. D'Ancona, Y. Shibata, On global solvability of nonlinear viscoelastic equations in the analytic category, *Math. Methods Appl. Sci.* **17**(1994), No. 6, 477–486. https://doi.org/10.1002/mma.1670170605; MR1274154

[14] M. del Pino, R. Manásevich, Global bifurcation from the eigenvalues of the p-Lapiacian, *J. Differential Equations* **92**(1991), 226–251. https://doi.org/10.1016/0022-0396(91)90048-E; MR1120904

[15] X. He, W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, *Nonlinear Anal.* **70**(2009), 1407–1414. https://doi.org/10.1016/j.na.2008.02.021; MR2474927

[16] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[17] Z. Liang, F. Li, J. Shi, Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31**(2014), 155–167. https://doi.org/10.1016/j.anihpc.2013.01.006; MR3165283

[18] Z. Liang, F. Li, J. Shi, Positive solutions of Kirchhoff type nonlocal elliptic equation: a bifurcation approach, *Proc. Roy. Soc. Edinburgh Sect. A* **147**(2017), No. 4, 875–894. https://doi.org/10.1017/S0308210516000378; MR3681567

[19] J. L. Lions, On some equations in boundary value problems of mathematical physics, in: *Contemporary developments in continuum mechanics and partial differential equations (Proceedings of the International Symposium on Continuum Mechanics and Partial Differential Equations)*, North-Holland Mathematics Studies, Vol. 30, North-Holland, Amsterdam, 1978, pp. 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3

[20] T. F. Ma, J. E. Muñoz Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, *Appl. Math. Lett.* **16**(2003), No. 2, 243–248. https://doi.org/10.1016/S0893-9659(03)80038-1; MR1962322

[21] A. Mao, Z. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, *Nonlinear Anal.* **70**(2009), No. 3, 1275–1287. https://doi.org/10.1016/j.na.2008.02.011; MR2474918

[22] K. Perera, Z. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differential Equations* **221**(2006), No. 1, 246–255. https://doi.org/10.1016/j.jde.2005.03.006; MR2193850

[23] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7**(1971), 487–513. https://doi.org/10.1016/0022-1236(71)90030-9; MR0301587

[24] J. Sun, C. Tang, Existence and multiplicity of solutions for Kirchhoff type equations, *Nonlinear Anal.* **74**(2011), No. 4, 1212–1222. https://doi.org/10.1016/j.na.2010.09.061; MR2746801

[25] L. Wen, X. Tang, S. Chen, Ground state sign-changing solutions for Kirchhoff equations with logarithmic nonlinearity, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 47, 1–13. https://doi.org/10.14232/ejqtde.2019.1.47; MR3991096

[26] Z. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, *J. Math. Anal. Appl.* **317**(2006), No. 2, 456–463. https://doi.org/10.1016/j.jmaa.2005.06.102; MR2208932